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ABSTRACT
This paper deals with the fractional generalization of the integro-differential diffusion–wave equation for the Heisenberg sub-Laplacian, with homogeneous Bitsadze–Samarskii type time-nonlocal conditions. For the considered problem, we show the existence, uniqueness and the explicit representation formulae for the solution.

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1. Introduction

The purpose of this paper is to study Bitsadze–Samarskii type nonlocal problem for the time-fractional diffusion–wave equation with the Heisenberg sub-Laplacian $\Delta_{\mathbb{H}^n}$ in the space variables.

In [1], Bitsadze and Samarskii established the solvability of the new class of nonlocal problems for the elliptic equations, which relate the values of the solution on parts of the boundary with its values inside the domain. Such problems are called the Bitsadze–Samarskii problems. For the motivation of studying the Bitsadze–Samarskii type nonlocal problems, we refer to [2–8] and references therein.

Certain types of physical problems can be modelled by heat and wave equations with Bitsadze–Samarskii type initial conditions. The time multi-point heat and wave problems can arise from studying the atomic reactors [9,10] and of some inverse heat conduction problems for determining the unknown physical parameters [11]. Well-posedness and numerical simulations of time multi-point heat and wave problems were studied in [9,10,12–15].

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The version of such equations on the Heisenberg group serves as a basic model for the analysis of the sub-elliptic diffusion and wave propagation models, providing new insights and techniques for the whole problem.

Thus we consider the fractional integro-differential diffusion–wave equation

$$D^{\alpha}_{+0,t}u(t,x) - I^{\beta}_{+0,t}\Delta_{\mathbb{H}^n}u(t,x) = f(t,x), \quad t > 0,\ x \in \mathbb{H}^n, \tag{1.1}$$

where $f(t,x)$ is a sufficiently smooth function, $I^{\beta}_{+0,t}$ is the Riemann–Liouville fractional integral of order $\beta > 0$ [16]

$$I^{\beta}_{+0,t}u(t,x) = \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}u(s,x)\,\mathrm{d}s,$$

and $\partial^{\alpha}_{+0,t}$ is the Riemann–Liouville fractional derivative of order $0 < \alpha \leq 2$ ([16]) defined as

$$D^{\alpha}_{+0,t}u(t,x) = \frac{\partial^{[\alpha]+1}}{\partial t^{[\alpha]+1}} I^{1+[\alpha]-\alpha}_{+0,t}u(t,x),$$

where $[\alpha]$ is the integer part of α.

When $1 < \alpha < 2, \beta = 0$, Equation (1.1) is the time-fractional wave equation and when $0 < \alpha < 1, \beta = 0$, Equation (1.1) is the time-fractional diffusion equation. When $\alpha = 2, \beta = 0$, it represents the classical wave equation, while if $\alpha = 1, \beta = 0$, it represents the classical diffusion equation.

Many mathematical formulations of physical phenomena contain integro-differential equations, these equations arise in many fields such as fluid dynamics, biological models and chemical kinetics. If $\alpha = 1$, Equation (1.1) describes the heat conduction with memory [17,18], and many authors studied the analogue problems [19–25].

1.1. Heisenberg group

Let $\mathbb{H}^n$ be the Heisenberg group, that is, the space $\mathbb{R}^{2n+1}$ endowed with the group law

$$\xi \circ \xi' = \left(x + x', y + y', s + s' + 2\sum_{i=1}^{n}(x_i y_i' - x_i' y_i)\right),$$

where $\xi = (z,s) = (x,y,s) = (x_1,\dots,x_n,y_1,\dots,y_n,s)$, $z = (x,s)$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, $s \in \mathbb{R}$, $n > 1$; $\xi' = (x',y',s') \in \mathbb{R}^{2n+1}$. This group multiplication endows $\mathbb{H}^n$ with a structure of a nilpotent Lie group. A family of dilations is defined as

$$\delta_\tau(x,y,s) = (\tau x, \tau y, \tau^2 s), \quad \tau > 0.$$

The homogeneous dimension with respect to these dilations is $Q = 2n + 2$. The left invariant vector fields on the Heisenberg group are

$$X_i = \frac{\partial}{\partial x_i} + 2y_i\frac{\partial}{\partial s}, \quad Y_i = \frac{\partial}{\partial y_i} - 2x_i\frac{\partial}{\partial s}, \quad i = 1,2,\dots,n.$$

The horizontal gradient is

$$\nabla_{\mathbb{H}^n} = (X_1,\dots,X_n,Y_1,\dots,Y_n).$$

Hence, the sub-Laplacian $\Delta_{\mathbb{H}^n}$ is denoted by

$$\Delta_{\mathbb{H}^n} = \sum_{i=1}^{n} \left(X_i^2 + Y_i^2\right) = \nabla_{\mathbb{H}^n} \cdot \nabla_{\mathbb{H}^n}.$$

The (Kaplan) distance function on $\mathbb{H}^n$ is given by

$$dist(\xi, \xi') = \left\{((x-x')^2 + (y-y')^2)^2 + (s - s' - 2(x \cdot y' - x' \cdot y)^2)\right\}^{1/4}, \quad \xi, \xi' \in \mathbb{H}^n.$$

If $\xi' = 0$, then the distance function is

$$dist(\xi) = \left\{(|x|^2 + |y|^2)^2 + s^2\right\}^{1/4} = \left\{|z|^4 + s^2\right\}^{1/4}, \quad |z| = \sqrt{|x|^2 + |y|^2}.$$

1.2. Group Fourier transform

We begin with a reminder of the definition of the group Fourier transform on the Heisenberg group (see many sources, but e.g. [26,27] for its use in similar contexts). For $f \in \mathcal{S}(\mathbb{H}^n)$ the group Fourier transform is defined as

$$\widehat{f}(\lambda) := \int_{\mathbb{H}^n} f(x) \pi_\lambda(x)^* \, \mathrm{d}x,$$

with the Schrödinger representations

$$\pi_\lambda : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

for all $\lambda \in \mathbb{R}_* := \mathbb{R} \setminus \{0\}$. The inverse group Fourier transform formula can be written as

$$f(x) = \int_{\lambda \in \mathbb{R}_*} \mathrm{Tr}\left[\widehat{f}(\lambda)\pi_\lambda(x)\right] |\lambda|^n \, \mathrm{d}\lambda,$$

where Tr is the trace operator. The Plancherel formula becomes

$$\|f\|^2_{L^2(\mathbb{H}^n)} = \int_{\lambda \in \mathbb{R}_*} \|\widehat{f}(\lambda)\|^2_{\mathrm{HS}[L^2(\mathbb{R}^n)]} |\lambda|^n \, \mathrm{d}\lambda,$$

where $\|\cdot\|^2_{\mathrm{HS}[L^2(\mathbb{R}^n)]}$ is the Hilbert–Schmidt norm on $L^2(\mathbb{R}^n)$. For more details on Plancherel formula and the Hilbert Schmidt norm, we refer to [28, Chapter 6, Proposition 6.2.7].

2. Main results

In this paper, we will work in the functional space $C([0,T]; L^2(\mathbb{H}^n))$ with the norm

$$\|u\|_{C([0,T];L^2(\mathbb{H}^n))} := \sup_{t \in [0,T]} \|u(t, \cdot)\|_{L^2(\mathbb{H}^n)}$$

for all $u \in C([0,T]; L^2(\mathbb{H}^n))$.

Problem 2.1: Assume that $f \in C([0,T]; L^2(\mathbb{H}^n))$. In a domain $\Omega = \{(t,x) : (0,T) \times \mathbb{H}^n\}$ consider Equation (1.1) with Bitsadze–Samarskii type time-nonlocal conditions

$$I_{+0,t}^{1+[\alpha]-\alpha} u(0,x) + \sum_{i=1}^{m} \mu_i I_{+0,t}^{2-\alpha} u(T_i, x) = 0, \; [\alpha] D_{+0,t}^{\alpha-1} u(0,x) = 0, \quad x \in \mathbb{H}^n, \tag{2.1}$$

where $m \in \mathbb{N}, \mu_i \in \mathbb{R}$, $0 < T_1 \le T_2 \le \cdots \le T_m = T$.

We seek a solution $t^{1+[\alpha]-\alpha} u \in C([0,T]; L^2(\mathbb{H}^n))$ of the problems (1.1) and (2.1) such that $D_{+0,t}^{\alpha} u \in C([0,T]; L^2(\mathbb{H}^n))$ and $\Delta_{\mathbb{H}^n} u \in C([0,T]; L^2(\mathbb{H}^n))$.

The condition (2.2) can be interpreted as a multi-point non-resonance condition. Note that a similar problem for the time-fractional multi-term diffusion–wave equation was investigated by the authors in [29].

Theorem 2.2: *Let $f \in C([0,T]; L^2(\mathbb{H}^n))$, $D_{+0,t}^{\alpha} f \in C([0,T]; L^2(\mathbb{H}^n))$, and assume that the conditions*

$$\left| 1 + \sum_{i=1}^{m} \mu_i T_i^{\alpha-[\alpha]-1} E_{\alpha+\beta,\alpha-1} \left(-|\lambda| \nu_l T_i^{\alpha+\beta} \right) \right| \ge M > 0 \tag{2.2}$$

hold for all $l \in \mathbb{N}^n$ (where M is a constant), where

$$\nu_l = \sum_{j=1}^{n} (2l_j + 1), \quad l = (l_1, \ldots, l_n) \in \mathbb{N}^n.$$

Then there exists a unique solution of Problem 2.1, *and it can be written as*

$$u(t,x) = \int_{\mathbb{R}_*} \mathrm{Tr}[\widehat{K}(t,\lambda) \pi_\lambda(x)] \, \mathrm{d}\lambda, \tag{2.3}$$

where

$$\widehat{K}(t,\lambda)_{l,k} = \widehat{F}(t,\lambda)_{l,k} - \frac{\sum_{i=1}^{m} \mu_i \widehat{F}(T_i,\lambda)_{l,k} t^{\alpha-[\alpha]-1} E_{\alpha+\beta,\alpha-1} \left(-|\lambda| \nu_l t^{\alpha+\beta} \right)}{1 + \sum_{i=1}^{m} \mu_i T_i^{\alpha-[\alpha]-1} E_{\alpha+\beta,\alpha-1} \left(-|\lambda| \nu_l T_i^{\alpha+\beta} \right)},$$

for all $\lambda \in \mathbb{R}_*$ and $k \in \mathbb{N}$. Here

$$\widehat{F}(t,\lambda)_{l,k} = \int_0^t s^{\alpha-1} E_{\alpha+\beta,\alpha} \left(-|\lambda| \nu_l s^{\alpha+\beta} \right) \widehat{f}(t-s,\lambda)_{l,k} \, \mathrm{d}s,$$

and

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}$$

is the Mittag–Leffler function [16].

2.1. Proof of Theorem 2.2

2.1.1. Proof of the existence result

Let us take the group Fourier transform from Section 1.2 with respect to $x \in \mathbb{H}^n$, that is,

$$D^{\alpha}_{+0,t}\widehat{u}(t,\lambda) + \sigma_{\Delta_{\mathbb{H}^n}}(\lambda) I^{\beta}_{+0,t}\widehat{u}(t,\lambda) = \widehat{f}(t,\lambda), \tag{2.4}$$

where $\sigma_{\Delta_{\mathbb{H}^n}}(\lambda)$ is the symbol of the Heisenberg sub-Laplacian. It has the following form:

$$\sigma_{\Delta_{\mathbb{H}^n}}(\lambda) = |\lambda|\mathbf{H}_\tau \equiv |\lambda|\left(-\Delta_\tau + |\tau|^2\right), \tag{2.5}$$

where $\mathbf{H}_\tau$ is a harmonic oscillator operator for the variable $\tau \in \mathbb{R}^n$. For more information about the operator $\mathbf{H}_\tau$, we refer to [28,30].

It is known that the operator $\mathbf{H}_\tau$ is essentially self-adjoint in $L^2(\mathbb{R}^n)$ with a discrete spectrum ν_l, $l = (l_1, \dots, l_n) \in \mathbb{N}^n$.

Corresponding to μ_l, the harmonic oscillator operator has the complete system of orthonormal eigenfunctions $\{e_l\}_{l\in\mathbb{N}}$ on $L^2(\mathbb{R}^n)$. They take the form

$$e_l(\tau) = \prod_{j=1}^{n} P_{l_j}(\tau_j) e^{-(-|\tau|^2/2)},$$

where $P_m(\cdot)$ is the Hermite polynomial of order m. That is,

$$P_m(t) = c_m e^{|t|^2/2}\left(t - \frac{d}{\mathrm{d}t}\right)^m e^{-(|t|^2/2)}, \quad t > 0, \quad c_m = 2^{-m/2}(m!)^{-1/2}\pi^{-1/4}.$$

For more details, see [30].

Consequently, Equation (2.4) can be rewritten as

$$D^{\alpha}_{+0,t}\widehat{u}(t,\lambda)_{l,k} + |\lambda|\nu_l I^{\beta}_{+0,t}\widehat{u}(t,\lambda)_{l,k} = \widehat{f}(t,\lambda)_{l,k}, \tag{2.6}$$

for all $\lambda \in \mathbb{R}_*$, and any $l, k \in \mathbb{N}$. Here

$$\widehat{u}(t,\lambda)_{l,k} = (\widehat{u}(t,\lambda)e_l, e_k)_{L^2(\mathbb{R}^n)}$$

and

$$\widehat{f}(t,\lambda)_{l,k} = (\widehat{f}(t,\lambda)e_l, e_k)_{L^2(\mathbb{R}^n)}.$$

According to [31], the solution for Equation (2.6) satisfying initial conditions

$$I^{2-\alpha}_{+0,t}\widehat{u}(0,\lambda)_{l,k} = C,\ [\alpha]D^{\alpha-1}_{+0,t}\widehat{u}(0,\lambda)_{l,k} = 0,$$

can be represented in the form

$$\widehat{u}(t,\lambda)_{l,k} = \int_0^t s^{\alpha-1}E_{\alpha+\beta,\alpha}\left(-|\lambda|\nu_l s^{\alpha+\beta}\right)\widehat{f}(t-s,\lambda)_{l,k}\,\mathrm{d}s + CE_{\alpha+\beta,\alpha-1}\left(-|\lambda|\nu_l T_i^{\alpha+\beta}\right). \tag{2.7}$$

Then, it is not difficult to show that the solutions of Equation (2.6) satisfying the following conditions:

$$I^{2-\alpha}_{+0,t}\widehat{u}(0,\lambda)_{l,k} + \sum_{i=1}^{m}\mu_i I^{2-\alpha}_{+0,t}\widehat{u}(T_i,\lambda)_{l,k} = 0,\ [\alpha]D^{\alpha-1}_{+0,t}\widehat{u}(0,\lambda)_{l,k} = 0, \tag{2.8}$$

can be represented in the form

$$\widehat{u}(t,\lambda)_{l,k} = \widehat{F}(t,\lambda)_{l,k} - \frac{\sum_{i=1}^{m} \mu_i \widehat{F}(T_i,\lambda)_{l,k} t^{\alpha-[\alpha]-1} E_{\alpha+\beta,\alpha-1}\left(-|\lambda|\nu_l t^{\alpha+\beta}\right)}{1+\sum_{i=1}^{m} \mu_i T_i^{\alpha-[\alpha]-1} E_{\alpha+\beta,\alpha-1}\left(-|\lambda|\nu_l T_i^{\alpha+\beta}\right)}, \tag{2.9}$$

where

$$\widehat{F}(t,\lambda)_{l,k} = \int_0^t s^{\alpha-1} E_{\alpha+\beta,\alpha}\left(-|\lambda|\nu_l s^{\alpha+\beta}\right) \widehat{f}(t-s,\lambda)_{l,k}\, ds.$$

Indeed, the formula (2.9) can be checked by the direct calculation from (2.7) under the conditions (2.8).

Now, applying the inverse group Fourier transform, we obtain the solution of Problem 2.1 in the form (2.3).

We note that the above expression is well defined in view of the non-resonance conditions (2.2). Finally, based on (2.9), we rewrite our formal solution as (2.3).

2.1.2. Convergence of the formal solution

Here, we prove convergence of the obtained integrals corresponding to functions $t^{1+[\alpha]-\alpha}u(x,t)$, $D^{\alpha}_{+0,t}u(x,t)$ and $\Delta_{\mathbb{H}^n}u(x,t)$. To prove the convergence, we use the estimate for the Mittag–Leffler function

$$\left|E_{\alpha,\beta}\left(z\right)\right| \le \frac{C}{1+|z|}. \tag{2.10}$$

Let us first prove the convergence of (2.3). From the estimate (2.10), we have the following inequalities:

$$\left|\widehat{F}(t,\lambda)_{l,k}\right| \le C_1 \frac{\left|\widehat{f}(t,\lambda)_{l,k}\right|}{1+|\lambda|\nu_l},$$
$$\left|E_{\alpha+\beta,\alpha-1}\left(-|\lambda|\nu_l t^{\alpha+\beta}\right)\right| \le \frac{C_2}{1+|\lambda|\nu_l t^{\alpha+\beta}}$$

for some constants $C_1, C_2 > 0$. Hence, from these estimates it follows that

$$\begin{aligned} |t^{1+[\alpha]-\alpha}\widehat{u}(t,\lambda)_{l,k}|^2 &\le \left|t^{1+[\alpha]-\alpha}\widehat{F}(t,\lambda)_{l,k}\right|^2 \\ &\quad + \frac{1}{M}\sum_{i=1}^{n} |\mu_i|^2 |\widehat{F}(T_i,\lambda)_{l,k}|^2 \left|E_{\alpha+\beta,\alpha-1}\left(-|\lambda|\nu_l t^{\alpha+\beta}\right)\right|^2 \\ &\le C\frac{\left|\widehat{f}(t,\lambda)_{l,k}\right|^2}{(1+|\lambda|\nu_l)^2} + C\frac{1}{M}\sum_{i=1}^{n}|\mu_i|^2 \frac{\left|\widehat{f}(T_i,\lambda)_{l,k}\right|^2}{(1+|\lambda|\nu_l)^2}\frac{1}{(1+|\lambda|\nu_l t^{\alpha+\beta})^2}. \end{aligned}$$

Thus, since for any Hilbert–Schmidt operator A one has

$$\|A\|_{\mathrm{HS}}^2 = \sum_{l,k} |(A\phi_l, \phi_k)|^2$$

for any orthonormal basis $\{\phi_1, \phi_2, \ldots\}$, then we can consider the infinite sum over l, k of the inequalities provided by (2.9). This gives

$$\|t^{1+[\alpha]-\alpha}\widehat{u}(t,\lambda)\|_{\mathrm{HS}}^2 \le C\|(1+\sigma_{\Delta_{\mathbb{H}^n}}(\lambda))^{-1})\widehat{f}(t,\lambda)\|_{\mathrm{HS}}^2, \tag{2.11}$$

since $\sup_{t\in[0,T]} 1/(1+|\lambda|\nu_l t^{\alpha+\beta}) = 1$. Thus integrating both sides of (2.11) against the Plancherel measure on $\mathbb{R}_*$ and using the Plancherel identity [28], we obtain

$$\|t^{1+[\alpha]-\alpha}u\|_{C([0,T];L^2(\mathbb{H}^n))} \le C\|(I+\Delta_{\mathbb{H}^n})^{-1}f\|_{C([0,T];L^2(\mathbb{H}^n))}$$

and

$$\|\Delta_{\mathbb{H}^n}u\|_{C([0,T];L^2(\mathbb{H}^n))} \le C\|f\|_{C([0,T];L^2(\mathbb{H}^n))}.$$

Since $f \in C([0,T]; L^2(\mathbb{H}^n))$, we get

$$\|t^{1+[\alpha]-\alpha}u(t,x)\|_{C([0,T];L^2(\mathbb{H}^n))} < \infty$$

and

$$\|\Delta_{\mathbb{H}^n}u(t,x)\|_{C([0,T];L^2(\mathbb{H}^n))} < \infty.$$

The convergence of the integral corresponding to $D^{\alpha}_{+0,t}u(x,t)$ can be shown in a similar way.

To show the uniqueness of the solution, let us assume that there are two different functions u and v satisfying Problem 2.1. Now we introduce a new function w as the difference of the solutions u and v, that is, $w := u - v$.

Indeed, w satisfies the homogeneous equation

$$D^{\alpha}_{+0,t}w(t,x) - I^{\beta}_{+0,t}\Delta_{\mathbb{H}}w(t,x) = 0, \quad t>0,\ x\in\mathbb{H}^n, \tag{2.12}$$

with boundary conditions

$$I^{2-\alpha}_{+0,t}w(0,x) + \sum_{i=1}^{m}\mu_i I^{2-\alpha}_{+0,t}w(T_i,x) = 0,\ [\alpha]D^{\alpha-1}_{+0,t}w(0,x) = 0, \quad x\in\mathbb{H}^n, \tag{2.13}$$

where $\mu_i \in \mathbb{R}$, $0 < T_1 \le T_2 \le \cdots \le T_m = T$.

Again, repeating the same arguments, for the solution w of the problems (2.12) and (2.13) we obtain the estimate

$$\|t^{1+[\alpha]-\alpha}w\|_{C([0,T];L^2(\mathbb{H}^n))} \le 0.$$

Thus $0 = w = u - v$. The proof is complete.

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